MATHEMATICS Proceedings A 89 (2), June 23, 1986

An elementary approach to unsmoothing over cubes

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Communicated by Prof. A.C. Zaanen at the meeting of December 16, 1985

1. INTRODUCTION

The smoothing $T_n f$ of a locally-integrable function $f:\mathbb{R}^n \to \mathbb{R}$ is defined by

$$(T_n f)(x) = 2^{-n} \int_{x+Q} f(y)dy, \text{ where } Q = [-1,1]^n.$$

The problem of unsmoothing over cubes consists in the following:

i) Characterize the kernel and the range of T_n;

ii) Find a right inverse R^* of T_n such that for smooth g, $R^*(g)$ is optimally smooth;

iii) Analyse T_n on various subspaces of $L_{\text{loc}}(\mathbb{R}^n)$.

Our approach to the unsmoothing problem is elementary, no Fourier analysis is used. Because the case $n=1$ is treated in the same way in [4], we only sketch the proofs. We consider only $n=2$, but of course our method works for all n. (The case $n=1$ and many other unsmoothing problems are considered by F. John [2]).

2. GENERAL PROPERTIES OF T_2

DEFINITION 2.1

A function $g:\mathbb{R}^2 \to \mathbb{R}$ is called locally absolutely continuous, if g may be

* Contribution to the research project Reconstruction, Ko 506/8-1, of the German Research Council (DFG), directed by Professor D. Kölzow, Erlangen.

written in the form

$$g(x,y)=\int_0^x\int_0^y h(s,t)dtds+\int_0^x k(s)ds+\int_0^y l(t)dt+c$$

where $h\in L_{\text{loc}}(\mathbb{R}^2)$, $k\in L_{\text{loc}}(\mathbb{R})$, $l\in L_{\text{loc}}(\mathbb{R})$, *and* $c\in\mathbb{R}$.

We shall denote by $AC_{\text{loc}}(\mathbb{R}^2)$ the space of all locally-absolutely continuous functions on $\mathbb{R}^2$. The following lemma is important for our approach.

LEMMA 2.2

Let $f,g\in L_{\text{loc}}(\mathbb{R}^2)$. *Then* $T_2f=g$ *iff the following holds:*

i) $g\in AC_{\text{loc}}(\mathbb{R}^2)$

ii) $4\dfrac{\partial^2 g}{\partial x\partial y}(x,y)=f(x+1,y+1)-f(x-1,y+1)-f(x+1,y-1)+f(x-1,y-1)$ *a.e.*

iii) $4\dfrac{\partial g}{\partial x}(x,0)=\int_{-1}^{1}f(x+1,t)dt-\int_{-1}^{1}f(x-1,t)dt$ *a.e.*

iv) $4\dfrac{\partial g}{\partial y}(0,y)=\int_{-1}^{1}f(s,y+1)ds-\int_{-1}^{1}f(s,y-1)ds$ *a.e.*

v) $g(0,0)=(T_2f)(0,0)$.

(*Line* ii) *needs some explanation: By* $\partial^2g/\partial x\partial y=h$ *a.e. we mean more exactly that there is a function* d *such that* $\partial g/\partial x\,(x,y)=d(x,y)$ *a.e. for each fixed* y *and* $\partial d/\partial y\,(x,y)=h(x,y)$ *a.e. with respect to two-dimensional measure.*)

PROOF. Let $f\in L_{\text{loc}}(\mathbb{R}^2)$. We define

$$c:=\tfrac{1}{4}\int_{-1}^{1}\int_{-1}^{1}f(s,t)dtds,$$

$$k(x):=\tfrac{1}{4}\Big(\int_{-1}^{1}f(x+1,t)dt-\int_{-1}^{1}f(x-1,t)dt\Big)\text{ a.e.,}$$

$$l(y):=\tfrac{1}{4}\Big(\int_{-1}^{1}f(s,y+1)ds-\int_{-1}^{1}f(s,y-1)ds\Big)\text{ a.e., and}$$

$$h(x,y):=\tfrac{1}{4}(f(x+1,y+1)-f(x+1,y-1)-f(x-1,y+1)+\\ +f(x-1,y-1))\text{ a.e.}$$

Further we define $\tilde g$ by:

$$\tilde g(x,y)=\int_0^x\int_0^y h(s,t)dtds+\int_0^x k(s)ds+\int_0^y l(t)dt+c.$$

We have $c\in\mathbb{R}$, $k\in L_{\text{loc}}(\mathbb{R})$, $l\in L_{\text{loc}}(\mathbb{R})$ and $h\in L_{\text{loc}}(\mathbb{R}^2)$ and hence $\tilde g\in AC_{\text{loc}}(\mathbb{R}^2)$. An easy calculation shows that $\tilde g(x,y)=T_2f(x,y)$ for all $(x,y)\in\mathbb{R}^2$. By the fundamental theorem of calculus we get for $g=T_2f$:

a) $\dfrac{\partial g}{\partial x}(x,y)=\int_0^y h(x,t)dt+k(x)$ a.e. for each y;

b) $\dfrac{\partial g}{\partial y}(x, y) = \int\limits_0^x h(s, y)ds + l(y)$ a.e. for each x;

c) $\dfrac{\partial}{\partial y}\left(\int\limits_0^y h(x, t)dt + k(x)\right)(x, y) = h(x, y)$ a.e.

with respect to two-dimensional measure (since $h(x, \cdot) \in L_{\text{loc}}(\mathbb{R})$ for almost all $x \in \mathbb{R}$). Hence for $g = T_2 f$ the statements i)–iv) hold. Also g is uniquely determined by these properties.

From this, we can prove injectivity results such as:

COROLLARY 2.3

T_2 is injective on $L_p(\mathbb{R}^2)$ for $1 \leq p < \infty$.

PROPOSITION 2.4

Let $g \in AC_{\text{loc}}(\mathbb{R}^2)$ such that $g(x, y) = 0$ if $x \leq c$ or $y \leq c$, where $c \in \mathbb{R}$. Then there is a unique $f \in L_{\text{loc}}(\mathbb{R}^2)$ such that $T_2 f = g$ and $f(x, y) = 0$ if $x \leq c+1$ or $y \leq c+1$. Further, f is given by

$$f(x, y) = 4 \frac{\partial^2}{\partial x \partial y} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g(x - 2n - 1, y - 2m - 1) \ (a.e.).$$

PROOF. The uniqueness of f (a.e.) follows from lemma 2.2. Thus, it is enough to check the equation $T_2 f = g$ for the proposed f. Of course, similar statements and reconstruction formulas hold, if $g(x, y) = 0$ for all $x \leq c$ or $y \geq c$ and so on. Now we represent $g \in AC_{\text{loc}}(\mathbb{R}^2)$ as a sum $g = g_1 + g_2 + g_3 + g_4$, where $g_i \in AC_{\text{loc}}(\mathbb{R}^2)$ and $g_1(x, y) = 0$ if $x \leq c_1$ or $y \leq c_1$, $g_2(x, y) = 0$ if $x \leq c_2$ or $y \geq c_2$ and so on. Further, we can choose $g_i \in C^k(\mathbb{R}^2)$ if $g \in C^k(\mathbb{R}^2)$ ($k \in \mathbb{N}$, $k \geq 2$). By this consideration and 2.4, the following is proved:

PROPOSITION 2.5

$$T_2(L_{\text{loc}}(\mathbb{R}^2)) = AC_{\text{loc}}(\mathbb{R}^2).$$

Moreover, there exists a right inverse R^* of T_2 such that $R^*(C^k(\mathbb{R}^2)) \subseteq$ $\subseteq C^{k-2}(\mathbb{R}^2)$ for $k \geq 2$.

3. SMOOTHING FOR L_1-FUNCTIONS

We already know that T_2 is injective on $L_1(\mathbb{R}^2)$. Now we want to characterize $T_2(L_1(\mathbb{R}^2))$. We begin with the simpler class $L_{\text{com}}(\mathbb{R}^2) = \{f \in L_1(\mathbb{R}^2)/f$ has compact support$\}$.

PROPOSITION 3.1

$T_2(L_{\text{com}}(\mathbb{R}^2)) = \{g \in AC_{\text{loc}}(\mathbb{R}^2)/g$ has compact support, $\sum_{m \in \mathbb{Z}} g(x, y + 2m) =$ $= H_1(x)$, and $\sum_{n \in \mathbb{Z}} g(x + 2n, y) = H_2(y)\}$.

PROOF. Let $f \in L_{\text{com}}(\mathbb{R}^2)$. Then $T_2 f = g \in AC_{\text{loc}}(\mathbb{R}^2)$ with compact support. Moreover, $\sum_{m \in \mathbb{Z}} g(x, y + 2m) = \frac{1}{4} \int_{x-1}^{x+1} \int_{\mathbb{R}} f(s, t)dtds$ does not depend on y and, similarly, $\sum_{n\in\mathbb{Z}} g(x+2n, y)$ does not depend on x. Conversely, let $g \in AC_{\mathrm{loc}}(\mathbb{R}^2)$ with these properties. Then, by proposition 2.4, there is a unique $f_1 \in AC_{\mathrm{loc}}(\mathbb{R}^2)$ with $T_2 f_1 = g$ and $f_1(x, y) = 0$ if $x \leq c+1$ or $y \leq c+1$ (for some $c \in \mathbb{R}$) and f_1 is given by the formula of 2.4. The analogs of 2.4 yield f_2, f_3, and f_4 with $T_2 f_i = g$ and $f_2(x, y) = 0$ if $x \leq c+1$ or $y \geq c+1$ and so on. The respective reconstruction formulas show that $f_1 = f_2 = f_3 = f_4$ which proves our statement.

PROPOSITION 3.2

$$T_2(L_1(\mathbb{R}^2)) = \{g \in AC_{\mathrm{loc}}(\mathbb{R}^2)/g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial^2 g}{\partial x \partial y} \in L_1(\mathbb{R}^2);$$

$$\sum_{k=0}^{\infty} \sum_{\max(n,m)=k} \frac{\partial^2}{\partial x \partial y} g(x-2n, y-2m) \text{ converges in } L_1^{SW}$$

(that is, outside $x > c$, $y > c$ for each c) and the sum is in $L_1(\mathbb{R}^2)\}$ and the inverse is given by

$$(T_2 g)(x, y) = 4 \frac{\partial^2}{\partial x \partial y} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g(x-2n-1, y-2m-1).$$

PROOF. Easy calculations using 2.2 show that for $f \in L_1(\mathbb{R}^2)$ the function $g = T_2 f$ has the stated properties and the given reconstruction formula holds. Conversely, if $g \in AC_{\mathrm{loc}}(\mathbb{R}^2)$ has those properties, define $f \in L_1(\mathbb{R}^2)$ by

$$f(x, y) = 4 \frac{\partial^2}{\partial x \partial y} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g(x-2n-1, y-2m-1).$$

Then $T_2 f = g$ which proves the statement.

REMARK 3.3

It is easy to see that $T_2^{-1}: T_2(L_1(\mathbb{R}^2)) \to L_1(\mathbb{R}^2)$ is not continuous when both spaces have L_1-topology. In other words, the unsmoothing problem is ill-posed for these topologies. For the numerical treatment of such problems one has to apply regularization techniques (see [1]). By means of proposition 3.2 one can prove continuity and well-posedness results for T_2 restricted on various subspaces of $L_1(\mathbb{R}^2)$.

LITERATURE

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